Linear Algebra with Dirac notation

Note Title 10/3/2010

Vectors are equivalent to states in quantum mechanics : $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \cdots$

scalars : $a, b, c,$: complex numbers

Vector addition and scalar multiplication follows the common rules such as

1. commutative-ness ; $|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$

2. associative-ness ; $|\alpha\rangle + (|\beta\rangle + |\gamma\rangle) = (|\alpha\rangle + |\beta\rangle) + |\gamma\rangle$

3. distributive-ness ; $a(|\alpha\rangle + |\beta\rangle) = a|\alpha\rangle + b|\beta\rangle$

4. associative-ness ; $a(b|\alpha\rangle) = (ab)|\alpha\rangle$

5. $|\alpha\rangle + |0\rangle = |\alpha\rangle$

6. $|\alpha\rangle + |-\alpha\rangle = |0\rangle$

7. $0|\alpha\rangle = |0\rangle$

8. $1 \cdot |\alpha\rangle = |\alpha\rangle$

9. $|-\alpha\rangle = (-1)|\alpha\rangle$

* linear combination of $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \cdots$ is

$$a|\alpha\rangle + b|\beta\rangle + c|\gamma\rangle + \cdots$$

* $|\lambda\rangle$ is linearly independent of the set $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \cdots$ if it cannot be written as a linear combination of them

Ex, $\hat{k}$ is linearly independent of $\hat{i}$ and $\hat{j}$ but any vector in the xy plane is linearly dependent on $\hat{i}$ and $\hat{j}$

* If every vector can be written as a linear combination of a set of vectors, this set is said to **span** the space.

* A set of linearly independent vectors that span the space is called a **basis**.

* The number of vectors in any basis is called the **dimension**.

* With a basis $\{|e_1\rangle, |e_2\rangle, \cdots, |e_n\rangle\}$, if

$$|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \cdots a_n|e_n\rangle,$$

$$|\alpha\rangle \leftrightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$a_1, a_2, \cdots, a_n$ are called **components**.

Then $|\alpha\rangle + |\beta\rangle \leftrightarrow \begin{pmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{pmatrix}$

$$c|\alpha\rangle \leftrightarrow \begin{pmatrix} ca_1 \\ \vdots \\ ca_n \end{pmatrix}$$

Inner Products

With orthonormal basis $\{|e_1\rangle, |e_2\rangle, \cdots, e_n\rangle\}$, that is $\langle e_i | e_j\rangle = \delta_{ij}$

$$|\alpha\rangle \leftrightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad |\beta\rangle \leftrightarrow \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$\langle\alpha| \equiv (|\alpha\rangle)^\dagger = (a_1^*, \cdots, a_n)$ is called **bra** vector

, while $|\alpha\rangle$ is called **ket** vector

And $\langle\alpha|\beta\rangle = (a_1^*, \cdots, a_n^*)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

$= a_1^* b_1 + a_2^* b_2 + \cdots a_n^* b_n$

This is called the inner product.

- $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$
- $\langle\alpha|\left(b|\beta\rangle + c|\gamma\rangle\right) = b\langle\alpha|\beta\rangle + c\langle\alpha|\gamma\rangle$
- $\|\alpha\| \equiv \sqrt{\langle\alpha|\alpha\rangle}$; **norm** of $|\alpha\rangle$
  $= \sqrt{|a_1|^2 + |a_2|^2 + \cdots |a_n|^2}$
- $a_i = \langle e_i|\alpha\rangle$

$*$ Analogy with ordinary vectors

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$

$\langle\alpha|\beta\rangle = \|\alpha\|\|\beta\| \cos\theta$? **No !**

But still, $|\langle\alpha|\beta\rangle|^2 = \|\alpha\|^2\|\beta\|^2 \cos^2\theta$

$\Leftrightarrow \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle = \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \cos^2\theta$

$\Rightarrow \sqrt{|\langle\alpha|\beta\rangle|^2} = \|\alpha\|\,\|\beta\| \cos\theta$

$\therefore |\langle\alpha|\beta\rangle|^2 \le \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$

Called Schwarz Inequality

* Gram - Schmidt Procedure to find an orthonormal basis

$\boxed{\text{Ex}_1}$ $|\psi_a\rangle = |\psi_1\rangle + 2i|\psi_2\rangle$

$|\psi_b\rangle = i|\psi_1\rangle + |\psi_2\rangle$

with $|e_1'\rangle \equiv \frac{|\psi_a\rangle}{\|\psi_a\|}$, find $|e_2'\rangle$

orthonormal to $|e_1'\rangle$ using the Gram-Schmidt procedure

Ans : $|e_2\rangle = |\psi_b\rangle - (\langle e_1'|\psi_b\rangle)|e_1'\rangle$

$$|e_2'\rangle = \frac{|e_2\rangle}{\|e_2\|}$$

$$|\psi_a\rangle = \begin{pmatrix} 1 \\ 2i \end{pmatrix} \Rightarrow \|\psi_a\| = \sqrt{5}$$

$$|\psi_b\rangle = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\Rightarrow |e_1'\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$|e_2\rangle = \begin{pmatrix} i \\ 1 \end{pmatrix} - \left[\frac{1}{\sqrt{5}}(1, -2i)\begin{pmatrix} i \\ 1 \end{pmatrix}\right] \cdot \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$= \begin{pmatrix} i \\ 1 \end{pmatrix} - \frac{1}{5}[i - 2i]\begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$= \begin{pmatrix} i \\ 1 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} -i \\ 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5i + i \\ 5 - 2 \end{pmatrix}$$

$$= \frac{3}{5}\begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

$$\Rightarrow |e_2'\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

$$\therefore \quad |e_1'\rangle = \frac{1}{\sqrt{5}}\left(|\psi_1\rangle + 2i\,|\psi_2\rangle \right)$$

$$|e_2'\rangle = \frac{1}{\sqrt{5}}\left(2i\,|\psi_1\rangle + |\psi_2\rangle \right)$$

Double check $\langle e_2'|e_1'\rangle = \frac{1}{5}(-2i, 1)\begin{pmatrix} 1 \\ 2i \end{pmatrix}$

$$= \frac{1}{5}(-2i + 2i) = 0$$

Matrices

Matrix is equivalent to operator in Quantum mechanics

$$|\beta\rangle = T|\alpha\rangle \iff \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} T_{11}, & T_{12}, & \cdots & T_{1n} \\ T_{21} & & \ddots & \\ \vdots & & \ddots & \\ T_{2n} & & & T_{nn} \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

, with $T_{ij} = \langle e_i | T | e_j \rangle$

* Transpose : $\tilde{T} \equiv \begin{pmatrix} T_{11}, & T_{21}, & \cdots & T_{2n} \\ T_{12} & & \ddots & \\ \vdots & & \ddots & \\ T_{1n} & & & T_{nn} \end{pmatrix}$

* Hermitian conjugate of " T " is

$$T^\dagger \equiv (\tilde{T})^* = \begin{pmatrix} T_{11}^*, & T_{21}^*, & \cdots & T_{2n}^* \\ T_{12}^* & & \ddots & \\ \vdots & & \ddots & \\ T_{1n}^* & & & T_{nn}^* \end{pmatrix}$$

This definition of hermitian conjugate applies to vectors as well so that

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \Rightarrow \langle\alpha| = [\widetilde{|\alpha\rangle}]^* = (a_1^* \dots a_n^*)$$

Thus

$$\langle\alpha|\beta\rangle = (a_1^* \dots a_n^*) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

* If $T^\dagger = T$, T is called hermitian

* Generally, matrix multiplication is not commutative, that is,

$$T \cdot S \neq S \cdot T$$

* Commutator is defined as

$$[S, T] = ST - TS$$

* $$(ST)^\dagger = T^\dagger S^\dagger$$

* Identy matrix $I \equiv \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$

* Inverse matrix T^{-1}; $T \cdot T^{-1} = T^{-1} \cdot T = I$

$T^{-1} = \dfrac{1}{\det T} \cdot \tilde{C}$, with C the matrix of cofactors

$\Rightarrow$ If $\det T = 0$, T^{-1} does not exist and T is called singular

* $$(ST)^{-1} = T^{-1} \cdot S^{-1}$$

$\boxed{\text{Ex.}}$ $$T = \begin{pmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ 2i & 0 & 1 \end{pmatrix}$$

$$\det T = 1 \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 2i & 1 \end{vmatrix} + i \begin{vmatrix} 0 & 2 \\ 2i & 0 \end{vmatrix}$$

$$= 2 + i(-4i) = 2 + 4 = 6$$

$$C = \begin{pmatrix} \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}, & -\begin{vmatrix} 0 & 0 \\ 2i & 1 \end{vmatrix}, & \begin{vmatrix} 0 & 2 \\ 2i & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & i \\ 0 & 1 \end{vmatrix}, & \begin{vmatrix} 1 & i \\ 2i & 1 \end{vmatrix}, & -\begin{vmatrix} 1 & 0 \\ 2i & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & i \\ 2 & 0 \end{vmatrix}, & -\begin{vmatrix} 1 & i \\ 0 & 0 \end{vmatrix}, & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & -4i \\ 0 & 1+2 & 0 \\ -2i & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -4i \\ 0 & 3 & 0 \\ -2i & 0 & 2 \end{pmatrix}$$

$$\therefore\ T^{-1} = \frac{\tilde{C}}{\det T} = \frac{1}{6} \begin{pmatrix} 2 & 0 & -2i \\ 0 & 3 & 0 \\ -4i & 0 & 2 \end{pmatrix}$$

Check $$\begin{pmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ 2i & 0 & 1 \end{pmatrix} \cdot \frac{1}{6} \begin{pmatrix} 2 & 0 & -2i \\ 0 & 3 & 0 \\ -4i & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 2+4, & 0, & -2i+2i \\ 0 & 6 & 0 \\ 4i-4i & 0 & 4+2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} //$$

Eigenvectors and eigenvalues

$T|\alpha\rangle$ is like rotating the vector $|\alpha\rangle$ and multiplying by a scalar onto it.

- Special set of vectors that remain unchanged in this process except for a constant factor are called "eigen vectors".

That is, if $T|\alpha\rangle = \lambda|\alpha\rangle$,

$|\alpha\rangle$ is called eigenvector and λ is called the eigenvalue corresponding to the eigenvector

* How to find $|\alpha\rangle$ and λ ?

$$T|\alpha\rangle = \lambda|\alpha\rangle \Rightarrow (T - I\cdot\lambda)|\alpha\rangle = 0$$

This will have non-zero solution for $|\alpha\rangle$ only if $\det(T - I\lambda) = 0$. Otherwise $T - I\lambda$ will have an inverse matrix and $|\alpha\rangle$ will be zero.

$\boxed{\text{Ex}_1}$ $T = \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}$

Find eigenvectors and eigenvalues.

$$\det \begin{vmatrix} 2-\lambda & 0 & -2 \\ -2i & i-\lambda & 2i \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left(-(i-\lambda)(1+\lambda)\right) - 2\left(-(i-\lambda)\right) = 0$$

$$\Rightarrow (i-\lambda)\left((2-\lambda)(1+\lambda) - 2\right) = 0$$

$$\Rightarrow (i-\lambda)\left(\cancel{2} - \lambda + 2\lambda - \lambda^2 - \cancel{2}\right) = 0$$

$$\Rightarrow (i-\lambda)(\lambda - \lambda^2) = 0$$

$$\Rightarrow \lambda(1-\lambda)(i-\lambda) = 0$$

$$\lambda = 0, 1, i$$

For $\lambda = 0$

$$\begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\Rightarrow \quad 2a - 2c = 0 \quad \Rightarrow \quad c = a$$

$$-2i \cdot a + ib + 2ic = 0$$

$$\Rightarrow -2a + b + 2a = 0 \quad \Rightarrow \quad b = 0$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

with normalization

For $\lambda = 1$

$$\begin{pmatrix} 2-1 & 0 & -2 \\ -2i & i-1 & 2i \\ 1 & 0 & -1-1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\Rightarrow \quad a - 2c = 0, \Rightarrow \quad a = 2c$$

$$-2ia + (i-1)b + 2ic = 0$$

$$\Rightarrow -4ic + (i-1)b + 2ic = 0$$

$$\Rightarrow b = \frac{2ic}{i-1} = \frac{2i(i+1)}{(i-1)(i+1)}c$$

$$= \frac{2(-1+i)}{-2}c = (1-i)c$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 1-i \\ 1 \end{pmatrix} c = \frac{1}{\sqrt{7}} \begin{pmatrix} 2 \\ 1-i \\ 1 \end{pmatrix}$$

with normalization

For $\lambda = i$

$$\begin{pmatrix} 2-i & 0 & -2 \\ -2i & 0 & 2i \\ 1 & 0 & -1-i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\Rightarrow \begin{aligned} &(2-i)a - 2c = 0 \\ &-2ia + 2ic = 0 \quad \Rightarrow \quad c = a \\ &a - (1+i)c = 0 \end{aligned}$$

$$\Rightarrow [(2-i) - 2]a = 0 \quad \Rightarrow \quad a = 0$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$ with normalization

If you use these eigen vectors as a new basis, in this new basis, the matrix appears as diagonalized.

In other words, with $|e_1\rangle = |\alpha_{\lambda=0}\rangle$

$|e_2\rangle = |\alpha_{\lambda=1}\rangle$ and $|e_3\rangle = |\alpha_{\lambda=i}\rangle$

as the new basis,

$$|\alpha_{\lambda=0}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\alpha_{\lambda=1}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and $|\alpha_{\lambda=i}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

And in this new basis

$$T_{old} = \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix} \Rightarrow T_{new} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

* Every hermitian matrix is diagonalizable